Spectral Methods for Time-Domain Analysis of High-Speed Interconnects

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Outline

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- Time-domain sensitivity analysis

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Introduction

High-speed interconnect modeling: SI and EMC issues

- Ringing, attenuation, signal delay, distortion
- Crosstalk
- Non-linear terminations
- Frequency dependent phenomena (conductor and dielectric losses)
- EM radiation and susceptibility
Open issues

What is needed? Time-domain solutions for MTLs which allows to

- incorporate non-linear terminations (realistic driver and receiver models)
- incorporate frequency dependent phenomena (conductor and dielectric losses)
- model EM-to-MTL coupling (susceptibility)
- perform time-domain sensitivity analysis (pre-layout design)
- generate parametric models (pre-layout design)
Examples of TLs

Signal and power TLs
Telegrapher’s equations: a brief review

Time-domain

\[
\frac{\partial}{\partial z} v(z, t) = -R' i(z, t) - L' \frac{\partial}{\partial t} i(z, t)
\]

\[
\frac{\partial}{\partial z} i(z, t) = -G' v(z, t) - C' \frac{\partial}{\partial t} i(z, t)
\]

Complex frequency-domain

\[
\frac{d}{dz} V(z, s) = -Z' I(z, s)
\]

\[
\frac{d}{dz} I(z, s) = -Y' V(z, s)
\]

\[
Z' = R' + j\omega L' \quad Y' = G' + j\omega C'
\]
Telegrapher’s equations: a brief review

\[
\frac{d}{dz} V(z, s) = -\left[ R'(s) + sL'(s) \right] I(z, s) = -Z'(s)I(z, s)
\]

\[
\frac{d}{dz} I(z, s) = -\left[ G'(s) + sC'(s) \right] V(z, s) = -Y'(s)V(z, s)
\]

Existing methods for time-domain solution

- Lumped ladder network (closed-form available ([1]))
- Modal approach (similarity transformation) [2]
- Method of characteristics [3]
- Matrix Rational Approximation [3]
- Moments matching [3]
- Vector Fitting techniques ([4])
Lumped ladder network [1]

Two solutions

- discretization of the line along the length (equivalent circuit; not efficient);
- polynomial approach (half-T ladder networks; efficient, input/output macromodel)

\[ \Delta z = \frac{\ell}{n}, \quad Z_1(s) = Z'(s)\Delta z, \quad Y_2(s) = Y'(s)\Delta z, \quad K(s) = Y_2(s)Z_1(s) \]
Lumped ladder network [1]

Analytical solution for node potentials and currents flowing in the longitudinal and transverse impedance [1]

\[ V_{\beta}(s) = \frac{P_b^{n-\beta} (K(s))}{P_b^n (K(s))} V_0(s) = \frac{\sum_{j=0}^{n-\beta} b_{j,n-\beta} K^j(s)}{\sum_{j=0}^{n} b_{j,n} K^j(s)} V_0(s) \]

\[ I_{\beta 1}(s) = \frac{1}{Z_1(s)} \frac{P_c^{n-\beta+1} (K(s))}{P_b^n (K(s))} V_0(s) = \frac{1}{Z_1(s)} \frac{\sum_{j=0}^{n-\beta+1} c_{j,n-\beta+1} K^{j+1}(s)}{\sum_{j=0}^{n} b_{j,n} K^j(s)} V_0(s) \]

\[ I_{\beta 2}(s) = Y_2 \frac{P_b^{n-\beta} (K)}{P_b^n (K)} V_0(s) = Y_2 \frac{\sum_{j=0}^{n-\beta} b_{j,n-\beta} K^j(s)}{\sum_{j=0}^{n} b_{j,n} K^j(s)} V_0(s) \]

\[ b_{i,j} = \begin{pmatrix} j + i \\ j - i \end{pmatrix} = \begin{pmatrix} j + i \\ 2j \end{pmatrix} \quad c_{i,j} = \begin{pmatrix} i + j + 1 \\ j - i \end{pmatrix} = \begin{pmatrix} i + j + 1 \\ 2j + 1 \end{pmatrix} \]
Lumped ladder network [1]

- \( A = \sum_{j=0}^{n} b_{j,n} K^j(s) = P^n_b(K(s)) \)
- \( B = \left( \sum_{j=0}^{n} c_{j,n} K^{j+1}(s) \right) \cdot Z_2(s) = P^n_c(K(s)) \cdot Z_2(s) \)
- \( C = \left( \sum_{j=0}^{n} c_{j,n} K^{j+1}(s) \right) \cdot Z_1^{-1}(s) = P^n_c(K(s)) \cdot Z_1^{-1}(s) \)
- \( D = \sum_{j=0}^{n-1} b_{j,n-1} K^j(s) = P^{n-1}_b(K(s)) \)

\[
Y = \begin{bmatrix}
(P^n_c(K(s)) \cdot Z_2(s))^{-1} P^{n-1}_b(K(s)) & -(P^n_c(K(s)) \cdot Z_2(s))^{-1} \\
-(P^n_c(K(s)) \cdot Z_2(s))^{-1} & (P^n_c(K(s)) \cdot Z_2(s))^{-1} P^n_b(K(s))
\end{bmatrix}
\]

Polynomials \( P^n_b(K) \) and \( P^n_c(K) \) can be factored

\[
P^n_b(K) = \prod_{j=1}^{n} (K(s) - u_{j,n} U) \quad P^n_c(K) = K \prod_{j=1}^{n-1} (K(s) - v_{j,n-1} U)
\]

\[
u_{j,n} = -4 \sin^2 \left( \frac{(2j - 1) \pi}{2(2n + 1)} \right) \quad v_{j,n} = -4 \sin^2 \left( \frac{j \pi}{(n + 1) \frac{1}{2}} \right)
\]

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Lumped ladder network [1]

\[
Y_{11}(s) = \left[ \left( \prod_{j=1}^{n-1} (K(s) - v_{j,n-1}U) \right) (Z_1(s)) \right]^{-1} \cdot \prod_{j=1}^{n-1} (K(s) - u_{j,n-1}U)
\]

\[
Y_{12}(s) = - \left[ \left( \prod_{j=1}^{n-1} (K(s) - v_{j,n-1}U) \right) (Z_1(s)) \right]^{-1}
\]

\[
Y_{21}(s) = Y_{12}(s)
\]

\[
Y_{22}(s) = Y_{11}(s)
\]

\textit{y parameters in a rational form} \implies \text{pole-residue form possible}
**Lumped ladder network [1]**

\[
\begin{align*}
\text{det} & \left[ \left( \prod_{j=1}^{n-1} (K(s) - v_{j,n-1}U) \right) (Z_1(s)) \right] = 0 \\
\downarrow
\end{align*}
\]

\[
\begin{align*}
\left[ \prod_{j=1}^{n-1} \text{det} (K(s) - v_{j,n-1}U) \right] \text{det} (Z_1(s)) = 0
\end{align*}
\]

Residues evaluation in a closed form ⇒ Spice equivalent circuit

\[
Y(s) = \sum_{k=1}^{n_p} \frac{R_k}{s - p_k} + d \quad \rightarrow \quad \text{pole pruning} \quad \rightarrow \quad Y(s) \approx \sum_{k=1}^{n_d} \frac{R_k}{s - p_k} + d
\]
Vector fitting [4]

\[ H(s) = f(s) \quad \Rightarrow \quad \text{pole-residue form} \quad H(s) \approx \sum_{k=1}^{n_p} \frac{R_k}{s - p_k} + d \]

Two-step poles-residue identification

\[ \theta^i(s) = \sum_{n=1}^{N} \frac{c_n^i}{s - q_n^i} + 1 = \prod_{n=1}^{n_p} \frac{(s - z_n^i)}{(s - q_n^i)} \]

where \( \{q_n^0\} \) are arbitrarily chosen.

1. \( \theta^i(s)H(s) \approx \sum_{n=1}^{n_p} \frac{\tilde{c}_n^i}{s - q_n^i} + \tilde{c}_\infty \quad \Rightarrow \quad \text{compute} \quad c_n^i, \tilde{c}_n^i, \tilde{c}_\infty \)
   
   The poles of \( H(s) \) are equal to the zeros of the \( \theta(s) \) function!!

   They can be computed as a linear eigenvalue problem. It usually converges in few iterations.

2. The residues are computed by a linear least square problem.
Port currents are treated as current sources

\[
\frac{d}{dz} V(z, s) = - \left[ R'(s) + sL'(s) \right] I(z, s) = -Z'(s)I(z, s)
\]

\[
\frac{d}{dz} I(z, s) = - \left[ G'(s) + sC'(s) \right] V(z, s) + I_s(z, s)
\]

\[
= -Y'(s)V(z, s) + I_s(z, s)
\]

\[
I_s(z, s) = I_0(s)\delta(z) + I_\ell(s)\delta(z - \ell)
\]

The 2\textsuperscript{nd} order differential problem becomes:

\[
\frac{d^2}{dz^2} V(z, s) - \gamma^2(s)V(z, s) = -Z'(s)I_s(z, s), \quad (\gamma^2(s) = Z'(s)Y'(s))
\]

with homogeneous boundary conditions:

\[
I(z, s)|_{z=0} = I(z, s)|_{z=\ell} = 0 \implies \frac{d}{dz} V(z, s)|_{z=0} = \frac{d}{dz} V(z, s)|_{z=\ell} = 0
\]
Telegrapher’s equations: the Green’s function method

Telegrapher’s equations as a vector Sturm-Liouville problem

\[ \left[ L + \lambda r(z) \right] y(z, s) = f(z, s) \]

with boundary conditions (Dirichlet or Neumann or mixed type)

\[ \alpha_1 y(z, s) + \alpha_2 \frac{d}{dz} y(z, s) \bigg|_{z=0} = 0; \quad \beta_1 y(z, s) + \beta_2 \frac{d}{dz} y(z, s) \bigg|_{z=\ell} = 0 \]

Multiconductor transmission lines:

- \( L = Ud^2/dz^2 \), \((U\) identity matrix)\)
- \( \lambda = -\gamma^2(s) = -Z'(s)Y'(s) \)
- \( r(z) = U \),
- \( f(z, s) = -Z'(s)I_s(z, s) \)
- \( y(z, s) = V(z, s) \)

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Telegrapher's equations: the Green's function method

Dyadic Green's function for a vector Sturm-Liouville problem

\[ y(z, s) = \int_0^\ell G(z, z', s) f(z', s) dz' \]

\[ G(z, z', s) = \sum_{j=1}^N G_j(z, z', s) u_j = \sum_{j=1}^N \sum_{i=1}^N G_{ij}(z, z', s) u_i u_j \]

where \( G_j(z, z', s), j = 1, \cdots, N \) must satisfy

\[ [L + \lambda r(z)] G_j(z, z', s) = \delta(z, z') u_j \quad j = 1, \cdots, N \]

+ homogeneous boundary conditions

\[ \alpha_1 G_j(z, z', s) + \alpha_2 \frac{d}{dz} G_j(z, z', s) |_{z=0} = 0 \]

\[ \alpha_1 G_j(z, z', s) + \alpha_2 \frac{d}{dz} G_j(z, z', s) |_{z=\ell} = 0 \]
Telegrapher’s equations: the Green’s function method

**Self-adjoint problem** $\implies$ spectral representation of the Green’s function

$$G_j(z, z', s) = \sum_{n=0}^{\infty} a_{nj}(z', s) \phi_n(z)$$

where $[L + \lambda_n r(z)] \phi_n(z) = 0$ with the same boundary conditions for $G_j(z, z', s)$.

Uniform MTLs: $L = U d^2/dz^2$ and $r(z) = U$ $\implies$ scalar eigenvalue problem

$$[L + \lambda_n] \phi_n(z) = 0 + \text{ homogeneous boundary conditions}$$

Eigenfunctions $\phi_n(z), n = 1, \ldots, \infty$ satisfy the orthogonality condition

$$\int_0^\ell \phi_m^H(z) r(z) \phi_n(z) dz = \int_0^\ell \phi_m(z) r(z) \phi_n(x) dz = \delta_{mn} U$$
Telegrapher’s equations: the Green’s function method

1. Enforce $G_j(z, z', s)$ to be the solution of the equation

$$[L + \lambda r(z)] G_j(z, z', s) = \delta(z, z')u_j + b.c.$$ 

2. Use the orthonormality condition

3. Obtain the vector of amplitude coefficients $a_{mj}(z', s)$

$$a_{mj}(z', s) = (\lambda - \lambda_m)^{-1} \phi_m(z')u_j \quad j = 1, \ldots, N$$

4. Obtain the matrix of amplitude coefficients $a_m(z', s)$

$$a_m(z', s) = (\lambda - \lambda_m)^{-1} \phi_m(z')$$

5. Obtain the dyadic Green’s function $G(z, z', s)$ as

$$G(z, z', s) = \sum_{n=0}^{\infty} a_n(z', s)\phi_n(z) = \sum_{n=0}^{\infty} (\lambda - \lambda_n)^{-1} \phi_n(z')\phi_n(z)$$
Computation of the eigenfunctions and eigenvalues

\[
\left[ \frac{d^2}{dz^2} + k_n^2 \right] \phi_n(z) = 0
\]

where \( k_n^2 = \lambda_n + \) homogenous boundary conditions of the Neumann type

\[
\frac{d}{dz} \phi_n(z) \bigg|_{z=0} = \frac{d}{dz} \phi_n(z) \bigg|_{z=\ell} = 0
\]

The solution is

\[
\phi_n(z) = A_n \cos(k_n x)
\]

with

- \( k_n = n\pi/\ell \) \( n = 0, 1, 2, \ldots \)
- \( A_0 = \sqrt{\frac{1}{\ell}} \)
- \( A_n = \sqrt{\frac{2}{\ell}}, \quad n = 1, \ldots, \infty \)

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Series form of the dyadic Green’s function

\[
G(z, z', s) = -\sum_{n=0}^{\infty} \left[ \gamma^2(s) + \left( \frac{n\pi}{\ell} \right)^2 U \right]^{-1} \cdot A_n^2 \cos \left( \frac{n\pi}{\ell} z \right) \cos \left( \frac{n\pi}{\ell} z' \right)
\]

\[
V_0(s) = \int_0^\ell G(0, z', s) \left( -Z'(s)I_s(z', s) \right) \, dz'
\]

\[
= G(0, 0, s) \left( -Z'(s)I_0(s) \right) + G(0, \ell, s) \left( -Z'(s)I_\ell(s) \right)
\]

\[
= \sum_{n=0}^{\infty} \left[ \gamma^2(s) + \left( \frac{n\pi}{\ell} \right)^2 U \right]^{-1} \cdot \left[ A_n^2 Z'(s)I_0(s) + A_n^2 \cos(n\pi) Z'(s)I_\ell(s) \right]
\]

\[
V_\ell(s) = \int_0^\ell G(\ell, z', s) \left( -Z'(s)I_s(z', s) \right) \, dz'
\]

\[
= G(\ell, 0, s) \left( -Z'(s)I_0(s) \right) + G(\ell, \ell, s) \left( -Z'(s)I_\ell(s) \right)
\]

\[
= \sum_{n=0}^{\infty} \left[ \gamma^2(s) + \left( \frac{n\pi}{\ell} \right)^2 U \right]^{-1} \cdot \left[ A_n^2 Z'(s)I_0(s) \cos(n\pi) + A_n^2 Z'(s)I_\ell(s) \right]
\]
Series form of the dyadic Green’s function

\[
\begin{bmatrix}
V_0(s) \\
V_\ell(s)
\end{bmatrix}
= \begin{bmatrix}
Z_{11}(s) & Z_{12}(s) \\
Z_{21}(s) & Z_{22}(s)
\end{bmatrix}
\cdot
\begin{bmatrix}
I_0(s) \\
I_\ell(s)
\end{bmatrix}
\]

\[
Z_{11}(s) = Z_{22}(s) = \sum_{n=0}^{\infty} \left[ \gamma^2(s) + \left( \frac{n\pi}{\ell} \right)^2 U \right]^{-1} \cdot A_n^2 Z'(s)
\]

\[
Z_{12}(s) = Z_{21}(s) = \sum_{n=0}^{\infty} \left[ \gamma^2(s) + \left( \frac{n\pi}{\ell} \right)^2 U \right]^{-1} \cdot A_n^2 Z'(s) \cos(n\pi)
\]

A rational model can be developed provided a rational representation of \(\gamma(s)\) and \(Z'(s)\).

Poles and residues computation

**FIPUL:** MTLs with frequency independent per-unit-length parameters

\[
Z'(s) = R'_0 + sL'_0 \\
Y'(s) = G'_0 + sC'_0
\]

\(Z'(s)\) and \(Y'(s)\) polynomial matrices

**FDPUL:** MTLs with frequency dependent per-unit-length parameters

\[
Z'(s) = R'_0 + sL'_0 + \sum_{q=1}^{P_Z} \frac{R_Z}{s - p_{q,Z}} = \frac{B_p(s)}{A_p(s)} \\
Y'(s) = G'_0 + sC'_0 + \sum_{q=1}^{P_Y} \frac{R_Y}{s - p_{q,Y}} = \frac{D_p(s)}{C_p(s)}
\]

\(Z'(s)\) and \(Y'(s)\) rational matrices via VF [4]
Poles and residues computation

- The poles of the transmission line can be evaluated as the zeros of the common polynomial at the denominator of impedances

\[ P_n(s) = \det \left[ \gamma^2(s) + \left( \frac{n \pi}{\ell} \right)^2 U \right] = 0 \]

- FIPUL-MTLs is a special case of FDPUL-MTLs.
- For FDPUL-MTLs, poles can be evaluated as the solution of the following equation:

\[ Q_n(s) = \det \left[ \begin{bmatrix} B_p(s) & D_p(s) + A_p(s)C_p(s) \left( \frac{n \pi}{\ell} \right)^2 U \end{bmatrix} = 0 \]
Poles and residues computation

- The location of poles on the complex plane doesn't depend on the number of modes.
- The computation of both poles and residues is extremely fast because they are evaluated by solving low order algebraic equations and computing low order rational functions, respectively.

Magnitude spectrum of the transfer impedance $Z_{21}$


Pole pruning and reduced order model

General remarks

• The poles of $Z$ matrix entries are computed by solving algebraic equations;

• they are shared by each $Z$ matrix entry (macromodel);

• they can be regarded as the nearly optimal poles with the only approximation caused by the rational form of the per unit length impedance $Z'(s)$ and admittance $Y'(s)$;

• their knowledge allows to select the dominant ones avoiding the cumbersome computation of the moments of the system;
Pole pruning and reduced order model

- Accuracy in the bandwidth $0 - \omega_{max}$ ($\omega_{max}$ corresponds to the angular frequency beyond which the power of the excitation is negligible)
- Poles beyond $\omega_{max}$ could have an impact below $\omega_{max}$
Pole pruning and reduced order model

Two step pole pruning

- Maximum angular frequency $\omega_{\text{max}}$ of interest (source).
- Poles within a given bandwidth $\xi \omega_{\text{max}}$ ($\xi > 1$) are selected.
- Poles with magnitude of the corresponding residue larger than a fixed threshold are retained.

Condition 1 reads:

$$|\text{Im}(p_k)| < \xi \omega_{\text{max}} = \omega_{\text{MOR}} \quad k = 1, \cdots, n_{\text{poles}}$$

Condition 2 applies to each impedance $Z_{ij}(s)$ matrix entry and requires that

$$|R_k(m, n)| > \varsigma \quad k = 1, \cdots, S_1$$

Poles selected for different indexes $(m, n)$ are collected together.
Reduced order rational model

\[ Z(s) = \sum_{k=1}^{\tilde{P}} \frac{R_k}{s - p_k} \]

- Conductor and dielectric losses are easily incorporated in the time-domain model
- Linear and non-linear terminations can be incorporated as additional equations (MNA stamps)
Realization

\[ V(s) = Z(s)I(s) \Rightarrow \text{State space representation} \]

\[
\frac{d}{dt} x(t) = Ax(t) + Bi(t) \\
v(t) = Cx(t) + Di(t)
\]

Standard realization procedures can be adopted to obtain the state space matrices \( A, B, C, D \).
Realization

Example: 2 ports; 2 poles: \((\tilde{P} = 2)\), number of states \(q = 2\):

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\cdot
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} =
\begin{bmatrix}
\tilde{P} \\
\sum_{m=1}^{\tilde{P}} \frac{\tilde{R}_{11m}}{s - \tilde{p}_m} + \\
\sum_{m=1}^{\tilde{P}} \frac{\tilde{R}_{12m}}{s - \tilde{p}_m}
\end{bmatrix}
\cdot
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]

which reads:

\[
\left\{
\begin{array}{l}
V_1 = \sum_{m=1}^{\tilde{P}} \tilde{R}_{11m} \cdot X_{2m-1}(s) + \sum_{m=1}^{\tilde{P}} \tilde{R}_{12m} \cdot X_{2m}(s) \\
V_2 = \sum_{m=1}^{\tilde{P}} \tilde{R}_{21m} \cdot X_{2m-1}(s) + \sum_{m=1}^{\tilde{P}} \tilde{R}_{22m} \cdot X_{2m}(s)
\end{array}
\right.
\]

where state variable \(X_{2m}\) have been used:
Realization

\[
\begin{align*}
X_1(s) &= \frac{I_1(s)}{s - p_1} \\
X_2(s) &= \frac{I_2(s)}{s - p_1} \\
\vdots \\
X_{2m-1}(s) &= \frac{I_1(s)}{s - p_m} \\
X_{2m}(s) &= \frac{I_2(s)}{s - p_m} \\
\vdots \\
X_{2\hat{p}-1}(s) &= \frac{I_1(s)}{s - p_{\hat{p}}} \\
X_{2\hat{p}}(s) &= \frac{I_2(s)}{s - p_{\hat{p}}}
\end{align*}
\]

\[
\begin{align*}
sX_1(s) &= p_1X_1(s) + I_1(s) \\
sX_2(s) &= p_1X_2(s) + I_2(s) \\
\vdots \\
sX_{2m-1}(s) &= p_mX_{2m-1}(s) + I_1(s) \\
sX_{2m}(s) &= p_mX_{2m}(s) + I_2(s) \\
\vdots \\
sX_{2\hat{p}-1}(s) &= p_{\hat{p}}X_{2\hat{p}-1}(s) + I_1(s) \\
sX_{2\hat{p}}(s) &= p_{\hat{p}}X_{2\hat{p}}(s) + I_2(s)
\end{align*}
\]
Realization

Inverse Laplace transform \[ \Rightarrow \]

\[
\begin{align*}
\frac{d}{dt} x_1(t) &= p_1 x_1(t) + i_1(t) \\
\frac{d}{dt} x_2(t) &= p_1 x_2(t) + i_2(t) \\
&\vdots \\
\frac{d}{dt} x_{2m-1}(t) &= p_m x_{2m-1}(t) + i_1(t) \\
\frac{d}{dt} x_{2m}(t) &= p_m x_{2m}(t) + i_2(t) \\
&\vdots \\
\frac{d}{dt} x_{2\bar{P}-1}(t) &= p_{\bar{P}} x_{2\bar{P}-1}(t) + i_1(t) \\
\frac{d}{dt} x_{2\bar{P}}(t) &= p_{\bar{P}} x_{2\bar{P}}(t) + i_2(t)
\end{align*}
\]
Realization

...finally the voltages are

\[
\begin{align*}
v_1(t) &= \sum_{m=1}^{\tilde{P}} \tilde{R}_{1m} \cdot x_{2m-1}(t) + \sum_{m=1}^{\tilde{P}} \tilde{R}_{12m} x_{2m}(t) \\
v_2(t) &= \sum_{m=1}^{\tilde{P}} \tilde{R}_{21m} \cdot x_{2m-1}(t) + \sum_{m=1}^{\tilde{P}} \tilde{R}_{22m} x_{2m}(t)
\end{align*}
\]

The previous equations can be recast as:

\[
\begin{align*}
\frac{d}{dt} x(t) &= A x(t) + B i(t) \\
v(t) &= C x(t)
\end{align*}
\]
Realization

where state spaces matrices can be built from poles and residues:

\[
A = \begin{bmatrix}
\tilde{p}_1 & 0 & \cdots & \cdots & 0 \\
0 & \tilde{p}_1 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & 0 \\
0 & 0 & \cdots & \tilde{p}_\tilde{\nu} & 0 \\
0 & 0 & \cdots & 0 & \tilde{p}_\tilde{\nu}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\vdots & \vdots \\
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
\tilde{R}_{11} & \tilde{R}_{12} & \cdots & \tilde{R}_{11\tilde{\nu}} & \tilde{R}_{12\tilde{\nu}} \\
\tilde{R}_{21} & \tilde{R}_{22} & \cdots & \tilde{R}_{21\tilde{\nu}} & \tilde{R}_{22\tilde{\nu}}
\end{bmatrix}
\]
Comparison with other MTL macromodeling techniques

- No similarity transform required
- No discretization of the line needed
- No IFFT required

and in addition

- moments computation (cumbersome) is completely avoided
- optimal rational macromodel for a fixed accuracy and bandwidth
- fast and accurate macromodelling $\Rightarrow$ well suited to be used within optimization processes
- refinement of a model only requires to sum additional modes (poles) to the macromodel (no re-evaluation needed, well suited for adaptive schemes)
Two-conductor transmission line with FIPUL parameters

\[ R' = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \Omega/m \]
\[ L' = \begin{bmatrix} 0.28 & 0.07 \\ 0.07 & 0.28 \end{bmatrix} \mu H/m \]
\[ C' = \begin{bmatrix} 0.122 & -0.05 \\ -0.05 & 0.122 \end{bmatrix} \text{nF/m} \]
\[ G' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{S/m} \]
Two-conductor transmission line with FIPUL parameters

- \( \ell = 5 \) cm
- \( R_s = R_{NE} = 50 \ \Omega, \ C_L = C_{FE} = 1.5 \ \text{pF} \)
- \( f_{\text{max}} = 10 \ \text{GHz}, \ n_{\text{modes}} = 40 \)

Pole location in the complex plane and residues of \( Z_{11} \)
Magnitude spectra of impedances $Z_{11}$ and $Z_{12}$ evaluated using the transmission line theory (TLT), the half-T ladder network (HTLN) and the proposed Green’s function based method (GF).
Load voltage (left) and far-end voltage (right). The port voltages are computed using the transmission line theory via IFFT (TLT), a standard halt-T ladder network (HTLN), the proposed Green’s function-based approach in the frequency domain via IFFT (GF-IFFT) and integrating the macromodel in the time domain (GF-TD).
Two-conductor transmission line with FDPUL parameters

- $\ell = 10 \text{ cm (Line 2 [6])}$
- $R_s = R_{NE} = 30 \ \Omega$, $C_L = C_{FE} = 1.5 \text{ pF}$
- $f_{\text{max}} = 10 \text{ GHz}$

Pole location in the complex plane and residues of $Z_{13}$
Magnitude and phase spectra of impedance $Z_{13}$. The exact impedance (TLT) is compared with the one obtained using the Green’s function-based approach (GF) and that obtained retaining only the dominant poles (GF-Mor).
Driven line voltages (left) and victim line voltages (right). The port voltages are computed using the transmission line theory via IFFT (TLT) and the proposed Green’s function-based approach by integration of the macromodel in the time domain (GF-TD).
Plane-Wave to Multiconductor Transmission Lines Coupling [7]

\[
E^i = E_0(s) \left( e_x a_x + e_y a_y + e_z a_z \right) e^{-s(k_x x + k_y y + k_z z)},
\]

\[
e_x = \sin \theta_E \sin \theta_p,
\]
\[
e_y = -\sin \theta_E \cos \theta_p \cos \phi_p - \cos \theta_E \sin \phi_p,
\]
\[
e_z = -\sin \theta_E \cos \theta_p \sin \phi_p + \cos \theta_E \cos \phi_p,
\]
\[
e_x^2 + e_y^2 + e_z^2 = 1.
\]
\[
\frac{d}{dz} V(z, s) = -Z'(s) I(z, s) + V'_F(z, s),
\]
\[
\frac{d}{dz} I(z, s) = -Y'(s) V(z, s) + I'_F(z, s),
\]
\[
V'_F(z, s) = \begin{bmatrix}
\vdots \\
-\frac{d}{dz} \int_{a}^{a'} E_i^t(\rho, z, s) \cdot t \, d\rho \\
+ E_i^i(x_k, y_k, z, s) - E_i^i(x_0, y_0, z, s) \\
\vdots 
\end{bmatrix},
\]
\[
I'_F(z, s) = -\left[ G'(s) + sC'(s) \right] \begin{bmatrix}
\vdots \\
\int_{a}^{a'} E_i^t(\rho, z, s) \cdot t \, d\rho \\
\vdots 
\end{bmatrix}.
\]
Plane-Wave to Multiconductor Transmission Lines Coupling [7]

\[
\frac{d^2}{dz^2} I(z, s) - \gamma^2(s) I(z, s) = -Y'(s)V'_F(z, s) + \frac{d}{dz} I'_F(z, s) = f(z, s)
\]

\[
\frac{dx(t)}{dt} = A x(t) + B i(t), \\
v(t) = C x(t) + D i(t) + \tilde{v}_F(t),
\]
Plane-Wave to Multiconductor Transmission Lines Coupling [7]

\[
[L + \lambda r(z)] G_I(z, z', s) = \delta(z, z'), \quad G_I(z, z', s) |_{z=0} = G_I(z, z', s) |_{z=\ell} = 0,
\]

\[
G_I(z, z', s) = - \sum_{n=0}^\infty C_n^2 \varphi_n(s) \chi_n(z) \chi_n(z'),
\]

\[
\varphi_n(s) = \left[ \gamma^2(s) + \left( \frac{n\pi}{\ell} \right)^2 U \right]^{-1}, \quad \chi_n(z) = \sin \left( \frac{n\pi}{\ell} z \right)
\]

\[
\gamma^2(s) = Y'(s)Z'(s) \quad \Rightarrow \quad \tilde{I}_F(z, s) = \int_0^\ell G_I(z, z', s)f(z', s)dz',
\]

where

\[
f(z, s) = -Y'(s)V'_F(z, s) + \frac{d}{dz}I'_F(z, s) = \Gamma(s)E_0(s)e^{-sk_z z},
\]

with \( \Gamma(s) = sY'(k_x x + k_y y)e_z \).
Plane-Wave to Multiconductor Transmission Lines Coupling [7]

\[ \tilde{I}_F(z, s) = \int_0^{\ell} G_I(z, z', s) f(z', s) dz' = \sum_{n=0}^{\infty} \left[ e^{-skz\ell} (-1)^n - 1 \right] \cdot N_n(s) \chi_n(z) E_0(s) \]

where

\[ N_n(s) = M_n(s) L_n(s), \quad M_n(s) = \frac{n\pi/\ell}{[s^2k_z^2 + (n\pi/\ell)^2]}, \]

\[ L_n(s) = H_n(s) \Gamma(s), \quad H_n(s) = \left[ \gamma^2(s) + \left( \frac{n\pi}{\ell} \right)^2 U \right]^{-1} \cdot C_n^2. \]

\[ \tilde{V}_F(z, s) = Y^{-1}(s) \left( I'_F(z, s) - \frac{d}{dz} \tilde{I}_F(z, s) \right) \]

\( \tilde{V}_F(z, s) \) and \( \tilde{I}_F(z, s) \) series of delayed rational functions.
Plane-Wave to Multiconductor Transmission Lines Coupling [7]

\[ \tilde{V}_F(0, s) = -E_{Ft}(0, s) + \sum_{n=0}^{\infty} Y^{-1}(s) N_n(s) \frac{n\pi}{\ell} \cdot \left[ 1 - (-1)^n e^{-skz} \right] E_0(s) \]

\[ \tilde{V}_F(\ell, s) = -E_{Ft}(\ell, s) + \sum_{n=0}^{\infty} Y^{-1}(s) N_n(s) \frac{n\pi}{\ell} \cdot \left[ (-1)^n - e^{-skz} \right] E_0(s) \]

\[ \dot{x}_1(t) = A_1 x_1(t) + B_1 e_0(t), \]
\[ \tilde{v}_{F,1}(t) = C_1 x_1(t) + D_1 e_0(t), \]
\[ \dot{x}_2(t) = A_2 x_2(t) + B_2 e_0(t), \]
\[ \tilde{v}_{F,2}(t) = C_2 x_2(t) + D_2 e_0(t), \]

Plane-Wave to Multiconductor Transmission Lines Coupling [7]

\[ e_0(t) = E_0 e^{-((t-t_0)/T)^2} \]

\[ E_0 = 1 \text{ kV/m}, \quad T = 0.4 \text{ ns} \]

\[ \theta_E = -\pi/2 \]

\[ \theta_p = \pi/3 \]

\[ \phi_p = -\pi/3 \]
Plane-Wave to Multiconductor Transmission Lines Coupling [7]

Conductor 1

Conductor 3
Sensitivity analysis

Voltage sensitivity with respect to a parameter $\lambda$

$$V = ZI \implies \hat{V}(z, s) = \hat{Z}(z, s) \begin{bmatrix} I_0(s) \\ I_d(s) \end{bmatrix} + Z(z, s) \begin{bmatrix} \hat{I}_0(s) \\ \hat{I}_d(s) \end{bmatrix}$$

$$\hat{Z}(z, s) = \sum_{n=0}^{\infty} \Psi_n(s) \frac{\partial \gamma^2(s)}{\partial \lambda} \Psi_n(s)^{-1} (-Z_s(s)) \phi_n(z) \otimes [1 \ (\ -1)^n]$$

$$+ \sum_{n=0}^{\infty} \Psi_n(s) \frac{\partial \gamma^2(s)}{\partial \lambda} \Psi_n(s)^{-1} (-\hat{Z}_s(s)) \phi_n(z) \otimes [1 \ (\ -1)^n]$$

$$Z(z, s) = \sum_{n=0}^{\infty} \Psi_n(s) \frac{\partial \gamma^2(s)}{\partial \lambda} \Psi_n(s)^{-1} (-Z_s(s)) \phi_n(z) \otimes [1 \ (\ -1)^n]$$

No similarity transform is needed!!

Time-domain sensitivity

\[
\begin{bmatrix}
\hat{V}_0(s) \\
\hat{V}_d(s)
\end{bmatrix} = 
\begin{bmatrix}
\hat{Z}_{11} & \hat{Z}_{12} \\
\hat{Z}_{21} & \hat{Z}_{22}
\end{bmatrix} \cdot 
\begin{bmatrix}
I_0(s) \\
I_d(s)
\end{bmatrix} + 
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix} \cdot 
\begin{bmatrix}
\hat{I}_0(s) \\
\hat{I}_d(s)
\end{bmatrix}
\]

\[
\dot{x}_1(t) = A_1 x_1(t) + B_1 i(t) \quad \dot{x}_2(t) = A_2 x_2(t) + B_2 \hat{i}(t)
\]

\[
\hat{v}_1(t) = C_1 x_1(t) + D_1 i(t) \quad \hat{v}_2(t) = C_2 x_2(t) + D_2 \hat{i}(t)
\]

\[
\hat{v}(t) = \hat{v}_1(t) + \hat{v}_2(t) = C_1 x_1(t) + C_2 x_2(t)
\]

\[
i(t) = i_s(t) - Gv(t) - C \frac{dv(t)}{dt} - f(v(t))
\]

\[
\hat{i}(t) = -G \hat{v}(t) - C \frac{d\hat{v}(t)}{dt} - \frac{df(v(t))}{dv(t)} \hat{v}(t)
\]

Incorporation of non-linear terminations is straightforward!!
Time-domain sensitivity

Coplanar microstrip transmission line (length $d = 7$ cm)

$L_{11} = L_{22} = 5.269 \cdot 10^{-7}$ H/m, $L_{12} = L_{21} = 1.389 \cdot 10^{-8}$ H/m,

$C_{11} = C_{22} = 5.058 \cdot 10^{-11}$ F/m, $C_{12} = C_{21} = -3.466 \cdot 10^{-13}$ F/m,

$R_{11} = R_{22} = 39.78$ [Ω/m],

$G_{11} = G_{22} = 2.576$ mS/m.

$i(t) = 0.001v^3(t)$

Parameterized models of MTLs [8]

**Parametric macromodeling of** $Z_{\text{pul}}(s, g)$ and $Y_{\text{pul}}(s, g)$

$$Z_{\text{pul}}(s, g) \simeq \tilde{Z}_{\text{pul}}(s, g) = \frac{N_{Z_{\text{pul}}}(s, g)}{D_{Z_{\text{pul}}}(s, g)} = \frac{\sum_{p=0}^{P_{Z_{\text{pul}}}} \sum_{v=0}^{V_{Z_{\text{pul}}}} c_{pv, Z_{\text{pul}}} \phi_p(s) \varphi_v(g)}{\sum_{p=0}^{P_{Z_{\text{pul}}}} \sum_{v=0}^{V_{Z_{\text{pul}}}} \tilde{c}_{pv, Z_{\text{pul}}} \phi_p(s) \varphi_v(g)}$$

$$Y_{\text{pul}}(s, g) \simeq \tilde{Y}_{\text{pul}}(s, g) = \frac{N_{Y_{\text{pul}}}(s, g)}{D_{Y_{\text{pul}}}(s, g)} = \frac{\sum_{p=0}^{P_{Y_{\text{pul}}}} \sum_{v=0}^{V_{Y_{\text{pul}}}} c_{pv, Y_{\text{pul}}} \phi_p(s) \varphi_v(g)}{\sum_{p=0}^{P_{Y_{\text{pul}}}} \sum_{v=0}^{V_{Y_{\text{pul}}}} \tilde{c}_{pv, Y_{\text{pul}}} \phi_p(s) \varphi_v(g)}$$

**Parametric macromodeling of modal impedances** $Z_n(s, g)$

$$Z_n(s, g) \simeq \tilde{Z}_n(s, g) = \frac{N_{Z_n}(s, g)}{D_{Z_n}(s, g)} = \frac{\sum_{p=0}^{P_{Z_n}} \sum_{v=0}^{V_{Z_n}} c_{pv, Z_n} \phi_p(s) \varphi_v(g)}{\sum_{p=0}^{P_{Z_n}} \sum_{v=0}^{V_{Z_n}} \tilde{c}_{pv, Z_n} \phi_p(s) \varphi_v(g)}$$

**Parametric macromodeling of modal impedances** $Z(s, g)$

$$Z(s, g) \simeq \tilde{Z}(s, g) = \frac{N_Z(s, g)}{D_Z(s, g)} = \frac{\sum_{p=0}^{P_Z} \sum_{v=0}^{V_Z} c_{pv, Z} \phi_p(s) \varphi_v(g)}{\sum_{p=0}^{P_Z} \sum_{v=0}^{V_Z} \tilde{c}_{pv, Z} \phi_p(s) \varphi_v(g)}$$
Parameterized models of MTLs [8]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (freq)</td>
<td>100 Hz</td>
<td>15 GHz</td>
</tr>
<tr>
<td>Spacing (S)</td>
<td>200 µm</td>
<td>400 µm</td>
</tr>
</tbody>
</table>

Open issues

- Power-lines over dissipative ground
- Parametric time-domain sensitivity
- Electrically long interconnects
- Non-uniform multiconductor transmission lines
- Spectral form of the propagation operator
Conclusions

The spectral Green’s function-based approach for the transient analysis of lossy and dispersive MTLs has been presented.

- Telegrapher’s equations are analytically solved as a self-adjoint Sturm-Liouville problem.

- The dyadic Green’s function and the $Z$ matrix of the MTL are computed in a series rational form.

- The rational macromodel is fast to be computed since the modes are orthogonal.

- Poles and residues can be easily identified and dominant poles selected.
Conclusions

- Conductor and dielectric frequency-dependent losses are easily incorporated in the time-domain macromodel.
- Plane wave-to-MTL coupling is exactly modeled in the time-domain.
- Port voltage and current time-domain sensitivities are easily computed.
- Parametric time-domain macromodels are efficiently generated.
- The numerical results have confirmed the accuracy of the proposed method to generate macromodels which are well suited for time-domain analysis.
References


Some recent publications
